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Double Integration and Its Applications in Some Problems in Physics and Engineering

Nguyen Duc Thuan[1*]
[1]Ta Thi Thanh Loan - Hung Vuong University, Phu Tho, Vietnam

Abstract: Integral integration is one of two basic and key calculations in the field of Mathematical Analysis. In particular, the concept of multiple integration is highly abstract, but has many applications in mathematics and other scientific fields. This article explores the application of double integrals in some physics and engineering problems.

Keywords: Applications of Double Integrals, Application in physics problems, Application in engineering problems.

Review Paper

***Corresponding Author:**
Nguyen Duc Thuan
Ta Thi Thanh Loan - Hung Vuong University, Phu Tho, Vietnam

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1. INTRODUCTION

Integration is not only a mathematical tool but also a key in many practical fields, from science, engineering to economics and medicine. This article deals with double integrals (double integrals) - a type of definite integral extended to functions with more than one real variable. Due to this extension, double integrals are highly abstract, but they have many applications, such as calculating the area of a flat region, the volume of a solid, the mass of a thin plate or calculating the area of a general curved surface... In physics, double integrals are often used to calculate quantities such as gravity, magnetic force, electric current through certain regions of space; they are also applied in determining the properties of flow and heat distribution in materials.

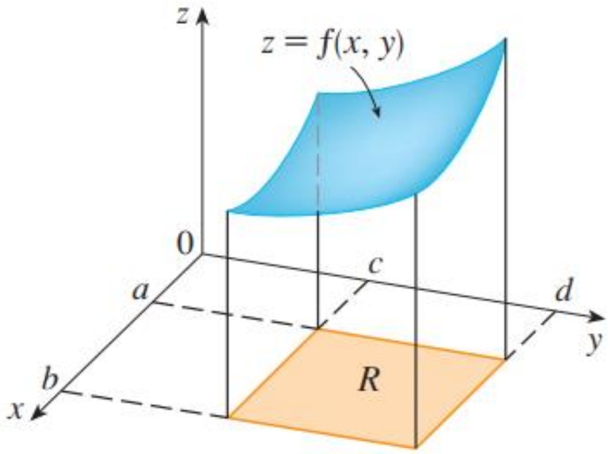
In engineering, we can use double integrals to design and analyze problems related to aerodynamics, fluid dynamics, and other fields where calculating quantities over complex shapes is necessary.

A double integral is a type of definite integral for functions with more than one variable. It extends the concept of integration of a function of one variable by calculating over a higher-dimensional space.

Below we will learn more about double integrals.

2. Double Integral

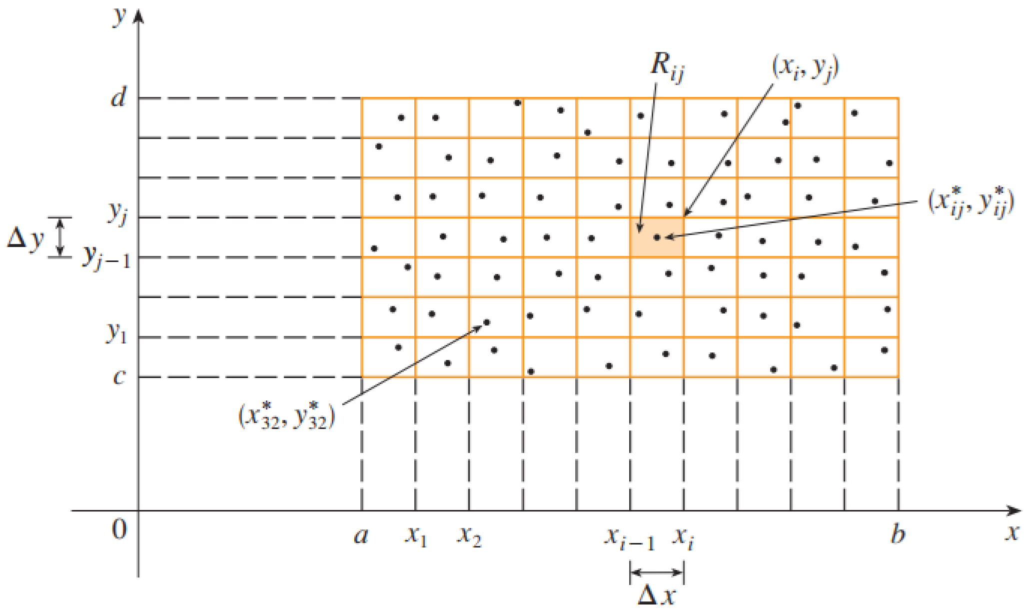
Extending from the definite integral, we consider a function of two variables f defined on a closed rectangular domain $R=[a,b]\times[c,d]=\left\{(x,y)\in \mathbb{R}^2 : a\le x\le b, c\le y\le d\right\}$, suppose $f(x,y)\ge 0$. The graph of f is a surface with equation $z=f(x,y)$. Let S be a solid lying above R and below the graph of f, that is: $S=\left\{(x,y,z)\in \mathbb{R}^3 : 0\le z\le f(x,y),(x,y)\in R\right\}$



We divide rectangle R into sub-rectangles:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] = \left\{ (x, y) : x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j \right\}$$

with the area of each rectangle being $\Delta A = \Delta x \Delta y$



Choosing a sample point (x^*_{ij}, y^*_{ij}) in each figure, we can approximate a part of S lying on each figure R_{ij} by a small rectangular box with a base R_{ij} and a height of $f(x^*_{ij}, y^*_{ij})$, the volume of this box is: $f(x^*_{ij}, y^*_{ij})\Delta A$, continuing this process will give an approximation of the volume for the entire part of S: $V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$. The volume is more accurate when m and n are larger, so we expect:

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$$

From there we have the definition of the double integral of the function f taken on the rectangle R as:

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$$

(if this limit exists)

To calculate a double integral, we usually start by integrating with respect to one variable while holding the other variable fixed and then integrating the result with respect to the second variable.

As mentioned above, the double integral has many important applications in mathematics and science, including physics and engineering problems. These applications will be explored in the following section.

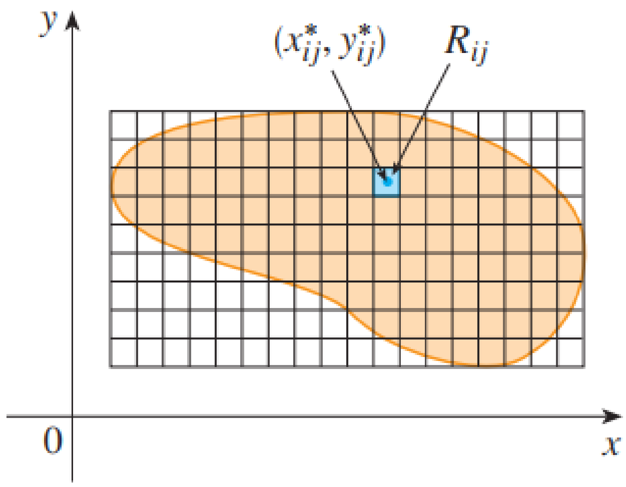
3. Some Applications of Double Integrals

a) Applications of Mass Calculation

When studying the application of definite integrals, we see that to calculate the moments and center of mass of a metal plate or thin plate, it can only be calculated in the case of constant density. However, with knowledge of double integrals, we can consider a thin plate with a changing density.

Suppose a thin plate occupies a region D in the (Oxy) plane, and its density at the point (x, y) in D is given by the formula: $\rho(x,y)=\lim\dfrac{\Delta m}{\Delta A}$, where $\rho(x,y)$ is a continuous function on D, Δm, ΔA and is the mass and area of a small rectangle containing (x, y), with the limit taken as the dimensions of the rectangle approach 0.

To find the mass m of the thin plate, we divide a rectangle R that contains D into many smaller rectangles R_{ij} of the same size.



If we choose an interior point (x^*_{ij}, y^*_{ij}) in R_{ij} , the mass of the slab R_{ij} will be approximately equal to $\rho(x^*_{ij}, y^*_{ij}).\Delta A$, where ΔA is the area of R_{ij} . Adding all the masses together, we get an approximation for the total mass, and then increasing the number of subrectangles, we get the total mass of the slab as the limiting value of the approximations, and it is the value of the double integral:

$$m=\lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}\rho(x^*_{ij}, y^*_{ij}).\Delta A=\iint_D \rho(x,y)dA.$$

Example 1: Calculate the mass of a circular slab with center $O(0,0)$ and radius R, knowing that the density is $\rho(x,y)=3\sqrt{x^2+y^2}$

Guide:

We have $M_D=\iint\limits_{x^2+y^2\le R^2} 3\sqrt{x^2+y^2}dxdy$. Convert to polar coordinates and calculate we get:

$$M_D=\iint\limits_{x^2+y^2\le R^2} 3\sqrt{x^2+y^2}dxdy=\int_0^{2\pi}d\varphi\int_0^R 3r^2dr=2\pi R^3$$

b) Application of Calculating Moment and Center of Mass

Similar to calculating mass, if using double integral, we can calculate the moment of a thin plate with changing density. Specifically:

The moment of the entire thin plate rotating around the Ox axis is: $M_{Ox} = \iint_D y\rho(x,y)dA$

The moment of the entire thin plate rotating around the Oy axis is: $M_{Oy} = \iint_D x\rho(x,y)dA$

We determine the center of mass $(\bar{x}, \bar{y})$ such that $m\bar{x} = M_{Oy};\ m\bar{y} = M_{Ox}$. That is, the plate will have the characteristic that all of its mass is concentrated at its center of mass. Therefore, the plate maintains horizontal equilibrium when supported at the center of mass.

Specifically, the thin plate has a density function of $\rho(x,y)$, then the coordinates of the center of mass are determined by the formula:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)dA;\ \ \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)dA$$

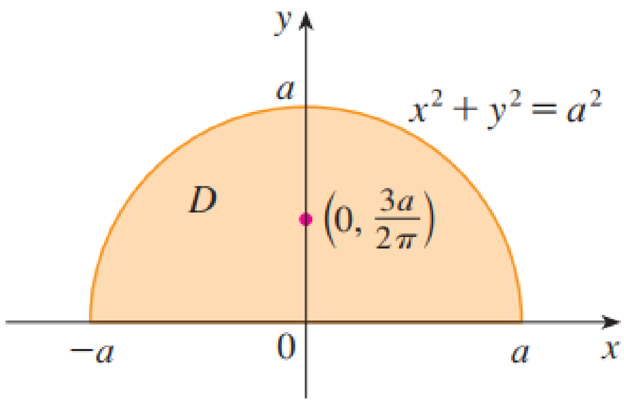
In which the mass: $m = \iint_D \rho(x,y)dA$

Example 2**:** The density at any point on a semicircular slab is proportional to its distance from the centre of the circle. Find the mass of the slab.

Guide:

Suppose the lamella is the upper half of the circle $x^2 + y^2 = a^2$

Then the distance from a point (x, y) to the center of the circle is $\sqrt{x^2+y^2}$. So the density function is $\rho(x,y) = K\sqrt{x^2+y^2}$ where K is a constant.



Easy to calculate: $m = \iint_D \rho(x,y)dA = \iint_D K\sqrt{x^2+y^2}dA = \int_0^{\pi}\int_0^{a} Kr.rdrd\theta = \frac{K\pi a^3}{3}$.

We see that the thin plate and the density function are both symmetric about the Oy axis, so the center of mass must lie on the Oy axis, that is $\bar{x} = 0$, $\bar{y}$ is calculated as follows:

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)dA = \frac{3}{K\pi a^3}\int_0^{\pi}\int_0^{a} r\sin\theta(Kr)rdrd\theta = \frac{3}{\pi a^3}\int_0^{\pi}\sin\theta d\theta\int_0^{a} r^3dr = \frac{3a}{2\pi}.$$

So, the center of mass of the thin plate lies at the point $\left(0;\dfrac{3a}{2\pi}\right)$.

c) Application of Calculating Moment of Inertia

Moment of inertia is a physical quantity characterizing the degree of inertia of objects in rotational motion.

For a particle with mass m moving around an axis, it is calculated by the formula mr^2, in which r is the distance from the particle to the axis of rotation. In the case of a thin plate with a density function $\rho(x,y)$

The moment of inertia of the thin plate rotating around the Ox axis is determined by $I_x = \iint\limits_D y^2\rho(x,y)dA$

The moment of inertia around the Oy axis is determined by $I_y = \iint\limits_D x^2\rho(x,y)dA$;

The moment of inertia around the origin is determined by $I_0 = \iint\limits_D (x^2+y^2)\rho(x,y)dA$

Example 3: Find the moments of inertia I_x, I_y, I_0 of a homogeneous disc D with density function $\rho(x,y)$, center at the origin, radius a.

Guide:

The boundary of D is a circle $x^2+y^2=a^2$. Converting to polar coordinates we get:

$$I_0 = \iint\limits_D (x^2+y^2)\rho(x,y)dA = \rho\int_0^{2\pi}\int_0^a r^2 r dr d\theta = \frac{\pi\rho a^4}{2}.$$

Due to the symmetry of the problem $I_x = I_y$, therefore $I_x = I_y = \dfrac{I_0}{2} = \dfrac{\pi\rho a^4}{4}$.

Remark: We know that the mass of the disk is (density multiplied by area). Therefore, the torque of the disk rotating around the origin can be written as:

$$I_x = \frac{\pi\rho a^4}{4} = \frac{1}{2}(\rho\pi a^2)a^2 = \frac{1}{2}ma^2.$$

Thus, if we increase the mass or radius of the disk, the moment of inertia will increase. In other words, the role of moment of inertia in rotational motion is similar to the role of mass in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

d) Applications in Engineering Problems

Double integrals are also used to solve problems related to dynamic forces in mechanical design, from calculating forces on structures to optimizing designs.

Example 4: Suppose we have a rectangular metal plate with length *a* and width *b*. Now we want to calculate the force acting on this metal plate due to a pressure evenly distributed $\rho(x,y)$ over the entire surface of the metal plate.

Guide: Assuming uniform pressure distribution is $\rho(x,y) = p_0$, where p_0 is a constant. The region of integration is the entire surface of the metal plate, the region D with $0 \le x \le a, 0 \le y \le b.$

$$F = \iint\limits_D \rho(x,y)dA = \iint\limits_D p_0 dxdy = p_0\int_0^b\int_0^a dxdy = p_0 ab.$$

Thus, the force acting on the metal plate is $F = p_0 ab.$

Comment: This example can be applied to the design of load-bearing metal plates in structures such as bridges, buildings, or machine parts. By calculating the forces, engineers can ensure that their designs are strong enough to withstand the actual stresses.

The 2-integral also has many other applications in engineering, such as in design optimization. For example, optimizing the design of an aircraft wing: in the design of an aircraft wing, it is important to optimize the shape of the wing to reduce drag and increase lift. The 2-integral can be used to calculate quantities such as drag and lift on the surface of the wing.

Suppose the airplane wing has a complex shape and is described by a function $f(x,y)$ defined on domain D. Then the lift force (L) can be calculated by the square integral of the pressure $\rho(x,y)$ on the wing surface: $D = \iint_D d(x,y)dxdy$. By changing the shape of the wing (i.e. changing the domain D and the function $f(x,y)$) we can optimize the above integrals to achieve maximum lift and minimum drag. In addition to some of the above applications, the double integral is also used to calculate the probability when two random variables appear.

In addition to some of the above applications, the two-layer integral is also used to calculate the probability when two random variables appear.

Consider a pair of continuous random variables X and Y. The joint probability density distribution function of X and Y is a function f of two variables such that the probability that (X, Y) lies on a domain D is:

$$P((X,Y) \in D) = \iint_D f(x,y)dA$$

In particular, if the domain is a rectangle, then the probability that X lies between a and b and the probability that it lies between c and d:

$$P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)dydx$$

This probability is the volume above the rectangle $D = [a,b] \times [c,d]$ and below the graph of the joint probability density distribution function.

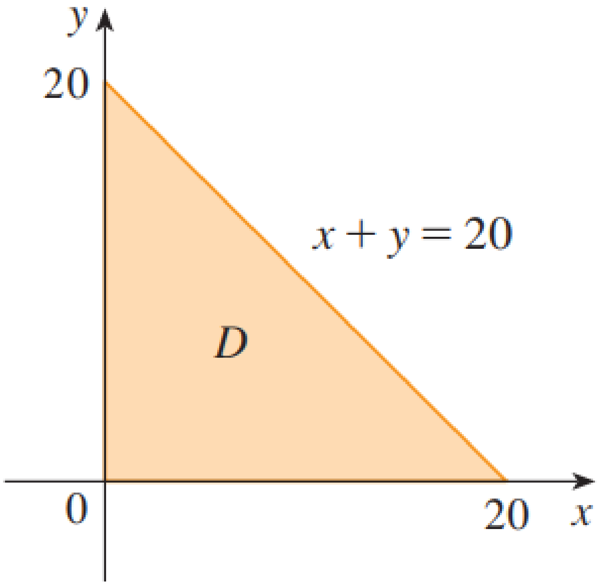
***Example 5*:** The management of a movie theater determines that the average time moviegoers wait in line to buy tickets for this week's movie is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming these wait times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before they are seated.

Guide: Assuming that both the waiting time X to buy popcorn and the waiting time Y to buy popcorn are modeled by exponential probability density functions, we can write their respective density functions as follows:

$$f_1(x) = \begin{cases} 0 & khi\ x < 0 \\ \frac{1}{10} e^{-x/10} & khi\ x \ge 0 \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0 & khi\ y < 0 \\ \frac{1}{5} e^{-y/5} & khi\ y \ge 0 \end{cases}$$

Since X and Y are independent of each other, the joint probability density distribution function is: $f(x,y) = f_1(x).f_2(y) = \frac{1}{50} e^{-x/10}.e^{-y/5} \quad khi\ x \ge 0, y \ge 0$ and 0 in the remaining cases.

The problem requires finding the probability that X + Y < 20: $P(X+Y<20) = P((X,Y) \in D)$, where D is a triangular region as shown in the figure:



Therefore we have:

$$P(X+Y<20)=\iint_D f(x,y)dA=\int_0^{20}\int_0^{20-x}\frac{1}{50}e^{-x/10}.e^{-y/5}\,dydx=1-e^{-4}-2e^{-2}\approx 0,7476.$$

This means that about 75% of moviegoers had to wait less than 20 minutes before getting into their movie seats.

4. CONCLUSION

In general, double integrals, also known as double integrals, are a powerful and important mathematical tool in mathematics, especially in multivariable calculus. They not only help solve complex problems but also have many applications in addressing practical issues in science and engineering. From calculating areas and volumes to application problems in physics and engineering. Thus, mastering double integrals not only helps us gain a better understanding of complex mathematical concepts but also opens up many opportunities to apply this theory in other scientific fields.

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